Journal of Computational and Applied Mathematics 20 (1987) 373–378
North-Holland

On the boundary element method with mesh refinement on curves with corners

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Received 28 May 1986
Revised 10 December 1986

Abstract: An indirect boundary integral formulation for the boundary value of the Laplacian will be given. This leads to the integral of the first kind. We study the problem on the boundary curve of some polygon. The Galerkin scheme for solving the boundary integral equation is analyzed.

For the approximation of the solution we use B-spline spaces on graded meshes, which are adapted to the known singularity of the boundary charge at the edges. We derive optimal order global error estimates in various Sobolev norms for the Galerkin solution. The numerical analysis is based on the uniqueness of he solution, a coersiveness inequality and the regularity properties of the solution.

Keywords: Boundary element method, singularities, mesh refinement, Galerkin-method, error analysis.

1. Introduction

In the focus of our consideration are the numerical approximation schemes for solving the potential problem

$$\begin{aligned} &\Delta u = 0 \quad \text{in } \Omega, \\ &u|_{\Gamma} = f \quad \text{on } \Gamma = \partial\Omega, \\ &u(z) = b \ln|z| - \tfrac{1}{2}\omega + \mathrm{O}(|1/z|), \quad z \to \infty \end{aligned} \tag{1.1}$$

on the polygon Ω by means of the boundary element methods. It is well known [4] that for given boundary data (f, b) the solution u has the integral representation

$$u(z) = -\frac{1}{2\pi}\int_{\Gamma} g(y) \ln|z-y| \,\mathrm{d}\gamma - \tfrac{1}{2}\omega, \tag{1.2}$$

where (g, ω) solves the integral equation

$$Sg(x) - \tfrac{1}{2}\omega := -\frac{1}{2\pi}\int_{\Gamma} g(y) \ln|x-y| \,\mathrm{d}y - \tfrac{1}{2}\omega = f(x),$$

$$\Lambda g := \int g(y) \,\mathrm{d}\gamma = b. \tag{1.3}$$

The standard procedure for finding the approximate solution employs the Galerkin method. We define the Galerkin approximate solution (g_h, ω_h) from the finite-dimensional boundary element space $S_h \times \mathbb{R}$ by

$$(Sg_h - \tfrac{1}{2}\omega_h \mid \phi)_0 = (f \mid \phi)_0, \qquad \Lambda g_h = b \tag{1.4}$$

for all $\phi \in S_h$.

For smooth boundaries the asymptotic error analysis is nowadays well established e.g. by Hsiao and Wendland [8]. In the case of the polygon boundary the solution g of the integral equation (1.3) has singularities at the corners even if the boundary data f is smooth. Thus singularities requires a special treatment. One can use special singular elements beside the standard boundary element spaces to obtain optimal order of convergence [4,6]. An other possibility to recover the optimal order of convergence is to refine the mesh near the singularities [9]. Here we give the theoretical justification for this approach. We use B-spline spaces on the graded meshes for the approximation of the solution. The refinement is adapted to the known singularity of the solution near the edges. The same kind of approach was applied by Chandler [3] to the second kind integral equations on polygonal boundaries. In this paper we present the asymptotic error analysis in Sobolev norms.

On the boundary Γ, which is a union of straight line segments Γ_j joining the corner points $\{z_j\colon j = 1, \ldots, J\}$, i.e. $\Gamma = \overline{\cup \Gamma_j}$, we define the Sobolev spaces $H^s(\Gamma)$ for $s \geqslant 0$ as trace spaces of $H^{s+1/2}(\Gamma)$, and for $s < 0$ by duality. The norm in $H^s(\Gamma)$ is denoted by $\| \cdot \|_{H^s(\Gamma)}$ or just by $\| \cdot \|_s$. For more complete description of these spaces we refer to [4].

2. The mapping properties

Here we present briefly the known mapping properties of the integral operator

$$\mathscr{A} := \begin{bmatrix} S & -\tfrac{1}{2}\mathrm{Id} \\ \Lambda & 0 \end{bmatrix} \tag{2.1}$$

in Sobolev spaces $\mathscr{H}^s(\Gamma) = H^s(\Gamma) \times \mathbb{R}$. The following theorem is quoted from [4]. By the Mellin-transform techniques they obtained:

Theorem 2.1. (a) *The integral operator* $\mathscr{A}\colon \mathscr{H}^s(\Gamma) \to \mathscr{H}^{s+1}(\Gamma)$ *is continuous for* $-\frac{3}{2} \leqslant s \leqslant \frac{1}{2}$. *Further it is an isomorphism for* $-\frac{1}{2} - \alpha < s < -\frac{1}{2} + \alpha$, *where* $\alpha = \min\{1/\alpha_j - 1, 1/(2-\alpha_j) - 1 \mid j = 1, \ldots, J\}$, $\alpha_j\pi$ *is the interior angle of the polygon at the edge*, $0 < \alpha_j < 2$, $\alpha_j \neq 1$.

(b) *For the operator* $\mathscr{A}$ *holds the Gårding inequality*

$$(\mathscr{A}G \mid G_0) \geqslant \beta \| G \|_{\mathscr{H}^{-1/2}(\Gamma)} + (\mathscr{K}G \mid G)_0 \tag{2.2}$$

for all $G \equiv (g, \omega) \in \mathscr{H}^{-1/2}(\Gamma)$ *and* $\mathscr{K}\colon \mathscr{H}^{-1/2}(\Gamma) \to \mathscr{H}^{1/2}(\Gamma)$ *is a compact operator.*

Remark. Theorem 2.1 serves the basis for stability of the considered approximation schemes. It is well known that a harmonic function behaves like $O(|z - z_j|^{\alpha_j + 1})$ near the corner z_j [6]. Using this analytical property of the boundary value problem (1.1) one deduces the regularity of the solution to the integral equation (1.3).

We define $Z^s(\Gamma)$, $s \geqslant \alpha + \frac{1}{2}$, to contain functions of the form

$$u(x) = u_0(x) + \sum_{j,k} c_{jk}\mu_{jk}(x) + \sum_{j,k} b_{jk}\nu_{jk}(x),$$

where $u_0 \in H^s(\Gamma)$ and the singular functions are ($k = 1, 2, 3, \dots$)

$$\mu_{jk}(x) = \begin{cases} |x - z_j|^{k/\alpha_j - 1}, & k/\alpha_j \notin \mathbb{N}, \\ \theta_j |x - z_j|^{k/\alpha_j - 1} + |x - z_j|^{k/\alpha_j - 1} \ln|x - z_j|, & k/\alpha_j \notin \mathbb{N}; \end{cases}$$

$$\nu_{jk}(x) = \begin{cases} |x - z_j|^{k/(2-\alpha_j) - 1}, & k/(2-\alpha_j) \notin \mathbb{N}, \\ |x - z_j|^{k/(2-\alpha_j) - 1} + |x - z_j|^{k/(2-\alpha_j) - 1}\chi_j \ln|x - z_j|, & k/(2-\alpha_j) \in \mathbb{N}, \end{cases}$$

when x is sufficiently close to z_j otherwise they are smooth functions. Here c_{jk}, b_{jk}, θ_j, χ_j are functions having constant values on Γ_{j-1} and Γ_j, possibly different ones These functions are zero when $k/\alpha_j > s + \frac{1}{2}$ and $k/(2 - \alpha_j) > s + \frac{1}{2}$. The norm in $Z^s(\Gamma)$ is defined by setting $\| u \|_{Z^s(\Gamma)} = \| u \|_s + \sum_{j,k} |c_{jk}| + |b_{jk}|$. Now the following holds [6]:

Theorem 2.2. *The operator* $\mathscr{A}$: $Z^s(\Gamma) \times \mathbb{R} \to \mathscr{H}^{s+1}(\Gamma)$ *is an isomorphism for all* $s \geqslant \alpha + \frac{1}{2}$.

3. The boundary element spaces

For the approximation method we define the grid points of the partition Δ on each line segment Γ_j by

$$z_{j,i} := \begin{cases} z_j + \frac{1}{2}(z_{j+1} - z_j)(i)^q([\tfrac{1}{2}N])^{-q}, & i = 0, \dots, [\tfrac{1}{2}N], \\ z_{j+1} + \frac{1}{2}(z_j - z_{j+1})(N - i)^q([\tfrac{1}{2}N])^{-q}, & i = [\tfrac{1}{2}N] + 1, \dots, N, \end{cases} \tag{3.1}$$

where $[\frac{1}{2}N] = \max\{ k \in \mathbb{N} \mid k \leqslant \frac{1}{2}N \}$ and $q > 0$ will be specified later. By $S_N^d(\Delta)$ denote the space of splines of order d i.e. each $\phi \in S_N^d(\Delta)$ is a polynomial of degree d on subinterval $I_{j,i} = [z_{j,i}, z_{j,i+1}]$ it is $(d-1)$-times continuously differentiable at each interior knots.

The main theorem of this chapter is the following.

Theorem 3.1. *Let* $q \geqslant (2(d+1)+1)/(1+2\alpha)$, *where* α *is as in Theorem* 2.1. *Then for every* $u \in Z^s(\Gamma)$ *there exists* $\phi \in S_N^d(\Delta)$ *such that for all* $t \leqslant s \leqslant d+1$ *and* $t \leqslant \alpha + \frac{1}{2}$ *holds*

$$\| u - \phi \|_t \leqslant cN^{t-s} \| u \|_{Z^s(\Gamma)}, \tag{3.2}$$

where the constant c *depends only on* q, d *and* α.

In the proof of Theorem 3.1 we need some results concerning the approximation of the singular functions appearing in the solution of the integral equation. By the coordinate transformation it is sufficient to consider only the approximation of the function $\mu(x) = x^\alpha$ on the unit interval $I_0 = [0, 1]$. We do not give the exact proofs here. They are presented elsewhere [11].

Lemma 3.2. *Let the grading exponent q be as above. Then there exists $v \in C^{d+1}(I_0)$ such that*

$$\| v - \mu \|_t \leqslant cN^{t-(d+1)}, \quad 0 \leqslant t \leqslant \alpha + \tfrac{1}{2}. \tag{3.3}$$

Here the constant c depends on d, q and α.

Using the local approximation properties of the smoothest spline functions [5] one can prove the following lemma.

Lemma 3.3. *Let the grading exponent q and $v \in C^{d+1}(I_0)$ be as in previous lemma. Then there exists $\phi \in S_N^d(\Delta)$ such that*

$$\| v - \phi \|_t \leqslant cN^{t-(d+1)}, \quad 0 \leqslant t \leqslant \alpha + \tfrac{1}{2}, \tag{3.4}$$

and the error constant $c = c(q, d, \alpha)$.

Remark. The same kind results were obtained already by Rice in [10], but he used piecewise polynomials which were only continuous at grid points.

Next we extend the approximation properties to hold also with respect to the Sobolev norms with negative indices. Here we utilize the simultaneous approximation results of [2], which states that for $t_0 < 0$ there exists a constant $c(t_0) > 0$ such that for $u \in H^s(I_0)$ there exists $\psi \in S_N^d(\Delta)$ possessing the estimate

$$\| \psi - u \|_t \leqslant c(t_0) N^t \inf_{\chi \in S_N^d} \| \chi - u \|_0. \tag{3.5}$$

for all $t_0 \leqslant t \leqslant 0$. Hence there exists $\phi \in S_N^d(\Delta)$ approximation to μ for which it holds

$$\| \mu - \phi \|_t \leqslant c(t_0, q, d, \alpha) N^{t-(d+1)}. \tag{3.6}$$

Now we are able to prove Theorem 2.1. For that let $u \in Z^s(\Gamma)$, $s \geqslant \frac{1}{2} + \alpha$, in other words

$$u = u_0 + \sum_{j,k} c_{jk} \mu_{jk} + \sum_{j,k} b_{jk} \nu_{jk}.$$

Because for negative indices ($t < 0$)

$$\| u \|_t \leqslant \sum_{j=1}^{J} \| u \|_{H^t(\Gamma_j)}$$

it suffices to prove the theorem only on each line segments. By previous lemmas

$$\inf_{\psi \in S_N^d(\Delta)} \| \mu_{jk} - \psi \|_{H^t(\Gamma_j)} \leqslant cN^{t-(d+1)}. \tag{3.7}$$

For ν_{jk} holds analogous estimate.

When approximating the regular part we can use the standard approximation property of the smoothest splines [1]: for all $t_0 \leqslant t \leqslant s \leqslant d+1$ and $t \leqslant \alpha + \frac{1}{2}$

$$\inf_{\psi \in S_N^d(\Delta)} \| u_0 - \psi \|_t \leqslant cN^{t-s} \| u_0 \|_{H^s(\Gamma_j)}. \tag{3.8}$$

Combining (3.7) and (3.8) we finally obtain the estimate (3.2).

4. The error analysis

The approximate solution to the integral equation (1.3) is sought from the finite dimensional space $H_N^d = S_N^d(\Delta) \times \mathbb{R}$. Obviously this space possesses the same approximation properties in $\mathscr{H}^s(\Gamma)$ as $S_N^d(\Delta)$ in Sobolev spaces $H^s(\Gamma)$. The approximation problem can now be written as to find $G_N \in H_N^d$ such that

$$(\mathscr{A}G_N \mid \Psi)_0 = (F \mid \Psi)_0 \tag{4.1}$$

for all $\Psi \in H_N^d$, where $F = (f, b) \in H^s(\Gamma) \times \mathbb{R}$ and $s \geqslant \frac{1}{2}$.

By the mapping properties of the integral operator $\mathscr{A}$ one concludes with the standard techniques of the Galerkin methods [1]:

Theorem 4.1. *The Galerkin equations* (4.1) *has a unique solution* $G_N = (g_N, \omega_N)$ *having the quasioptimality condition*

$$\| g - g_N \|_{-1/2} + | \omega - \omega_N | \leqslant c \inf_{\phi \in S_N^d} \| g - \phi \|_{-1/2}. \tag{4.2}$$

Utilizing the approximation properties of the boundary element spaces presented previous chapter and the Aubin–Nitsche duality argument one finally obtains:

Theorem 4.2. *The solution* G_N *of the Galerkin equations* (4.1) *satisfies the asymptotic error estimates, provided the grading* $q \geqslant (2(d+1)+1)/(1+2\alpha)$,

$$\| g - g_N \|_t + | \omega - \omega_N | \leqslant cN^{t-s}(\| g \|_{Z^s(\Gamma)} + | \omega |), \tag{4.3}$$

where $-d-2 \leqslant t \leqslant s \leqslant d+1$, $t \leqslant -\frac{1}{2}$ *and* $s > \frac{1}{2} + \alpha$. *When* $-\frac{1}{2} \leqslant s < \alpha + \frac{1}{2}$ *the norm is replaced by* $\| \cdot \|_{H^s(\Gamma)}$ *on the right hand side.*

Proof. Let then $V \in Z^{d+1}(\Gamma) \times \mathbb{R}$ be such that

$$\| G - G_N \|_{\mathscr{H}^{-d-2}(\Gamma)} = (G - G_N \mid \mathscr{A}V)_0$$

and $\| \mathscr{A}V \|_{H^{d+1}(\Gamma) \times \mathbb{R}} = 1$. This holds by the duality of the Sobolev spaces. Using the orthogonality of the Galerkin-method for all $\chi \in H_N^d$ holds

$$\begin{aligned} \| G - G_N \|_{\mathscr{H}^{-d-2}(\Gamma)} &= |(\mathscr{A}G - \mathscr{A}G_N \mid V - \chi)_0| \\ &\leqslant c \| G - G_N \|_{\mathscr{H}^{-1/2}(\Gamma)} \| V - \chi \|_{\mathscr{H}^{-1/2}(\Gamma)}. \end{aligned} \tag{4.4}$$

By Theorem 4.1 and the approximation properties the right hand side can be estimated as

$$\| G - G_N \|_{\mathscr{H}^{-d-2}(\Gamma)} \leqslant cN^{-(s+d+2)}.$$

Here the constant c depends only on the grading exponent q, order of the splines d and on the parameter α. Analogously as above we can prove the statement for all $-d-2 \leqslant t \leqslant -\frac{1}{2}$. $\square$

As a further application we may now derive the error estimates to the approximate solution of the potential problem (1.1) given by the integral representation

$$u_N(z) = -\frac{1}{2\pi} \int_\Gamma g_N(y) \ln |z - y| \, d\gamma - \tfrac{1}{2}\omega_N. \tag{4.5}$$

By the generalized Schwartz inequality we get the pointwise interior estimate

$$\begin{aligned} |u(z)-u_N(z)| &\leqslant \| \ln |z-y| \|_{d+2} \| g_N - g \|_{-d-2} + \tfrac{1}{2} |\omega - \omega_N| \\ &\leqslant cN^{-d-2-s} d(z, \Gamma)^{-1} \{ \| g \|_{Z^s(\Gamma)} + |\omega| \} \\ &\quad + cN^{-d-2-s} (\| g \|_{Z^s} + |\omega|), \end{aligned} \tag{4.6}$$

where we have used Theorem 4.2 to estimate $\| g - g_N \|_{-d-2}$ and $|\omega - \omega_N|$.

Acknowledgements

I am very grateful to Prof. Jukka Saranen of the University of Oulu for his advice and encouragement.

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